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Sen's theorem for hierarchies

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Abstract

Sen's theorem, which highlights the conflict between the Pareto condition and individual rights, is a fundamental result in the area of social choice. In this paper, we give an analog of Sen's theorem for consensus functions on hierarchies by showing that in almost all cases it is not possible to find a ternary Pareto consensus function where two distinct algorithms will have the same decisive impact on the consensus output.

1. Introduction

In 1970, Amartya Sen proved that there does not exist a Paretian social welfare function that can accommodate individual rights in a meaningful way [8]. Sen considered the situation where $k \geq 2$ individuals rank $n \geq 3$ alternatives such that the resulting rankings are linear orders on the set of alternatives. In this context, a social welfare function takes as input k individual rankings and outputs a single social ranking of the alternatives. (Sen's result only depends on the social outcome being an acyclic relation.) In this discussion we will assume that the domain of a social welfare function is unrestricted. The Pareto condition states that society ranks a over b if every individual ranks a over b. Sen modeled individual rights by introducing an axiom called *minimal liberalism*. This axiom says that there exist two individuals i and j and two pairs of alternatives $\{a, b\}$ and $\{x, y\}$ such that society's ranking of $\{a, b\}$ agrees with $i's$ ranking of $\{a, b\}$ and society's ranking of $\{x, y\}$ agrees with $j's$ ranking of $\{x, y\}$. So i and j have the right to decide for society how to rank $\{a, b\}$ and $\{x, y\}$, respectively. Sen's theorem, restricted to social welfare functions, states that there does not exist a social welfare function that satisfies the

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Pareto condition and the axiom of minimal liberalism. Developments of Sen's theorem in social choice theory can be found in Jerry Kelly's book [4].

The conflict between individual rights and the Pareto condition for social welfare functions translates in a natural way to consensus functions on hierarchies. Informally, a hierarchy (on a set S) is a type of classification scheme for the data set S and is often the application of a clustering algorithm to S. In this context, an individual is identified with a particular clustering algorithm and a consensus function takes as input a k-tuple of hierarchies and outputs a single consensus hierarchy. For more information about various types of consensus functions on hierarchies and consensus functions on other types of discrete structures see [3].

It is known that a hierarchy on S can be thought of as a special type of ternary relation on S [2] (see also [5]). Using the ternary approach, the Pareto condition for social welfare functions translates to a ternary Pareto condition for consensus functions on hierarchies. Individual rights for consensus functions are based on a notion of decisiveness for ordered triples. We will require that there exist two distinct individuals (algorithms) and two ordered triples such that one individual is decisive on the first triple while the second individual is decisive on the other ordered triple. The idea is that two individuals will have the same decisive impact on the consensus output. We show that, unless the two triples overlap in a particular way, there is a conflict between individual decisiveness and the ternary Pareto condition. The definitions of ternary Pareto and decisiveness used in this paper were defined and used in a previous paper [1] in order to obtain a version of Arrow's theorem for consensus functions on hierarchies.

In the next section we establish the notation and terminology related to the study of hierarchies. The final section contains our analog of Sen's theorem.

2. Terminology and notation

Let S be a set with $n \geq 5$ elements. A *hierarchy* on S is a collection H of nonempty subsets of S such that $S \in H$, $\{x\} \in H$ for all $x \in S$, and $A \cap B \in \{A, B, \emptyset\}$ for all $A, B \in H$. The set of all hierarchies on S is denoted by $\mathcal{H}$. A set X in a hierarchy H for which $1 < |X| < n$ is called a *nontrivial cluster* of H. The notation $H_\emptyset$ will be used to denote the hierarchy with no nontrivial clusters. For any nontrivial subset X of S let $H_X = H_\emptyset \cup \{X\}$. So H_X is a hierarchy on S where X is the only nontrivial cluster. Similarly, the notation $H_{X,Y}$ and $H_{X,Y,Z}$ will be used to represent hierarchies where X, Y and X, Y, Z are the only nontrivial clusters, respectively.

For each hierarchy H there is an associated ternary relation r_H on S where $(a, b, c) \in r_H$ if and only if there exists $X \in H$ such that $a, b \in X$ and $c \notin X$. This relation formalizes the notion that a and b are more similar to each other than either element is to c with respect to the hierarchy H. Thus, we will often write $ab \mid_H c$ instead of $(a, b, c) \in r_H$. In general, an ordered triple (a, b, c) of distinct elements from S is called a *triad*. The function that maps a hierarchy H on S to the ternary relation r_H is injective [5]. In fact, a subset X of S belongs to H if and only if $(a, b, c) \in r_H$ for all $a, b \in X$ and $c \notin X$. By identifying H with r_H, we see that a hierarchy is just a collection of triads. The following five properties of r_H are easy to prove and will be useful in the sequel. If a, b, c, d are four distinct elements from S, then

$$ab \mid_H c \;\Rightarrow\; ba \mid_H c \tag{1}$$

$$ab \mid_H c \;\Rightarrow\; ac \mid_H b \text{ fails} \tag{2}$$

$$ab \mid_H c \text{ and } ac \mid_H d \;\Rightarrow\; ab \mid_H d \tag{3}$$

$$ab \mid_H c \text{ and } ad \mid_H b \;\Rightarrow\; ad \mid_H c \tag{4}$$

$$ab \mid_H c \text{ and } ad \mid_H c \;\Rightarrow\; bd \mid_H c. \tag{5}$$

A *consensus function* is a map $C : \mathcal{H}^k \to \mathcal{H}$ where $k \geq 2$. Elements of $\mathcal{H}^k$, the k-fold Cartesian product, are called *profiles* and the conventional notation for profiles is $P = (H_1, \ldots, H_k)$, $P' = (H'_1, \ldots, H'_k)$, and so on. The hierarchy $C(P)$ will be referred to as the *consensus output*. If $K = \{1, \ldots, k\}$ and $I \subseteq K$, we say that I is *decisive* for (a, b, c) if for any profile P, $ab\,|_{C(P)}\, c$ whenever $ab\,|_{H_i}\, c$ for all $i \in I$. If K is decisive for all possible triads, then we say that C is a *ternary Pareto* consensus function. The ternary Pareto condition reflects true consensus; if each input hierarchy contains the triad (a, b, c), then the consensus output should also contain (a, b, c).

A more controversial idea is the requirement that one or more individuals should have a direct impact on some portion of the consensus result. If $C(P) = H_1$ for all $P = (H_1, \ldots, H_k)$, then C is a ternary Pareto consensus function where $\{1\}$ is decisive for all possible triads. In this example, $\{1\}$ has too much control. To avoid this situation we want to find a ternary Pareto consensus function where at least two individuals (algorithms) will have the same impact on the consensus result. Toward this end, we will require that there exist $i \neq j$ in K and triads (a, b, c) and (x, y, z) such that $\{i\}$ is decisive for (a, b, c) and $\{j\}$ is decisive for (x, y, z). We will see that in almost all cases it is not possible to maintain such a balance and still require that the consensus function satisfies the ternary Pareto condition.

3. The main result

We are now ready to state and prove our analog of Sen's theorem for the consensus function on hierarchies.

Theorem 1. *There exists a ternary Pareto consensus function $C : \mathcal{H}^k \to \mathcal{H}$ with $\{i\}$ decisive for (a, b, c), $\{j\}$ decisive for (x, y, z), and $i \neq j$ if and only if $\{a, b\} = \{x, y\}$.*

Proof. Assume that there exists a ternary Pareto consensus function $C : \mathcal{H}^k \to \mathcal{H}$ with $\{i\}$ decisive for (a, b, c), $\{j\}$ decisive for (x, y, z), $i \neq j$ and $\{a, b\} \neq \{x, y\}$. The triads (a, b, c) and (x, y, z) each contain three distinct elements but the sets $\{a, b, c\}$ and $\{x, y, z\}$ may overlap. If $\{a, b, c\} = \{x, y, z\}$, then consider the profile $P = (H_1, \ldots, H_k)$ where $H_i = H_{\{a,b\}}$ and $H_r = H_{\{x,y\}}$ for all $r \neq i$ in K. In particular, $H_j = H_{\{x,y\}}$. Notice that $ab\,|_{C(P)}\, c$ and $xy\,|_{C(P)}\, z$ by the decisiveness of $\{i\}$ and $\{j\}$, respectively. So $C(P)$ contains clusters A and B such that $a, b \in A$, $c \notin A$, $x, y \in B$, and $z \notin B$. Since $\{a, b, c\} = \{x, y, z\}$ and $\{a, b\} \neq \{x, y\}$ it follows that $A \cap B \notin \{A, B, \emptyset\}$ contrary to $C(P)$ being a hierarchy.

Now consider the case where $|\{a, b, c\} \cap \{x, y, z\}| = 2$. If $\{a, b, c\} \cap \{x, y, z\} = \{a, b\}$, then, since $\{a, b\} \neq \{x, y\}$ and $(u, v, w) \in r_H$ if and only $(v, u, w) \in r_H$ for any triad (u, v, w) and hierarchy H, we may assume $y = a$ and $z = b$. Let $P = (H_1, \ldots, H_k)$ be the profile where $H_i = H_{\{a,b\},\{c,x\}}$ and $H_r = H_{\{x,a,c\}}$ for all $r \neq i$ in K. Then $ab\,|_{C(P)}\, c$ and $ax\,|_{C(P)}\, b$ by the decisiveness of $\{i\}$ and $\{j\}$. Also, $xc\,|_{C(P)}\, b$ by ternary Pareto. By Eqs. (4) and (5), $ab\,|_{C(P)}\, c$ and $xa\,|_{C(P)}\, b$ implies that $xb\,|_{C(P)}\, c$. Thus $xb\,|_{C(P)}\, c$ and $xc\,|_{C(P)}\, b$ contrary to Eq. (2).

At this stage we know that $\{a, b, c\} \cap \{x, y, z\} \neq \{a, b\}$. A symmetric argument shows that $\{a, b, c\} \cap \{x, y, z\} \neq \{x, y\}$. Given Eq. (1) we may assume that $b \notin \{x, y, z\}$ and $y \notin \{a, b, c\}$. This leads to two possibilities: $a = x$ and $c = z$ or $a = z$ and $c = x$. If $a = x$ and $c = z$, then define $P = (H_1, \ldots, H_k)$ by $H_i = H_{\{a,b\},\{c,y\}}$ and $H_r = H_{\{x,y,c\},\{y,c\}}$ for all $r \neq i$. If $a = z$ and $c = x$, then define $P = (H_1, \ldots, H_k)$ by $H_i = H_{\{a,b,y\}}$ and $H_r = H_{\{x,y,b\},\{y,b\}}$ for all $r \neq i$ in K. For both possibilities decisiveness gives $ab\,|_{C(P)}\, c$ and $xy\,|_{C(P)}\, z$. Notice that $xy\,|_{C(P)}\, z$ is either $ay\,|_{C(P)}\, c$ or $cy\,|_{C(P)}\, a$. By Eq. (5), $ay\,|_{C(P)}\, c$ and $ab\,|_{C(P)}\, c$ implies that $by\,|_{C(P)}\, c$. But for the first profile, $cy\,|_{C(P)}\, b$ contrary to $by\,|_{C(P)}\, c$. For the second profile, $yc\,|_{C(P)}\, a$

and $yb \,|_{C(P)}\, c$ by ternary Pareto. By Eq. (4), $yc \,|_{C(P)}\, a$ and $yb \,|_{C(P)}\, c$ implies that $cb \,|_{C(P)}\, a$ contrary to $ab \,|_{C(P)}\, c$.

For the remainder of the proof we will invoke the five conditions given in Section 2 when needed without explicit references.

Now consider the case where $|\{a, b, c\} \cap \{x, y, z\}| = 1$. Given Eq. (1) and symmetry we only need to consider three possibilities: $a = x$ or $a = z$ or $c = z$. If $a = x$, then define $P = (H_1, \ldots, H_k)$ by $H_i = H_{\{a,b\},\{c,y,z\}}$ and $H_r = H_{\{a,y\},\{a,c,y,z\}}$ for all $r \neq i$ in K. Decisiveness of $\{i\}$ and $\{j\}$ implies that $ab \,|_{C(P)}\, c$ and $ay \,|_{C(P)}\, z$. By ternary Pareto, $yz \,|_{C(P)}\, b$ and $cy \,|_{C(P)}\, b$. Observe that $ab \,|_{C(P)}\, c$ and $ay \,|_{C(P)}\, z$ implies that $by \,|_{C(P)}\, z$ or $by \,|_{C(P)}\, c$ contrary to $yz \,|_{C(P)}\, b$ and $cy \,|_{C(P)}\, b$. If $a = z$, then define $P = (H_1, \ldots, H_k)$ by $H_i = H_{\{y,a,b\},\{y,a\},\{c,x\}}$ and $H_r = H_{\{a,x,y,c\},\{x,y,c\}}$ for all $r \neq i$ in K. Decisiveness gives $ab \,|_{C(P)}\, c$ and $xy \,|_{C(P)}\, a$. By ternary Pareto, $xc \,|_{C(P)}\, b$ and $ay \,|_{C(P)}\, b$. Now $xy \,|_{C(P)}\, a$ and $ay \,|_{C(P)}\, b$ implies that $xy \,|_{C(P)}\, b$. Next, $xy \,|_{C(P)}\, b$ along with $xc \,|_{C(P)}\, b$ implies that $yc \,|_{C(P)}\, b$. Notice that $yc \,|_{C(P)}\, b$ and $ay \,|_{C(P)}\, b$ implies that $ac \,|_{C(P)}\, b$ contrary to $ab \,|_{C(P)}\, c$. Finally, if $c = z$, then define $P = (H_1, \ldots, H_k)$ by $H_i = H_{\{a,b,x\},\{c,y\}}$ and $H_r = H_{\{x,y,a\},\{x,y,a,c\}}$ for all $r \neq i$ in K. As usual, decisiveness yields $ab \,|_{C(P)}\, c$ and $xy \,|_{C(P)}\, c$. By ternary Pareto, $ax \,|_{C(P)}\, c$. Now $ax \,|_{C(P)}\, c$ and $xy \,|_{C(P)}\, c$ implies that $ay \,|_{C(P)}\, c$. Next, $ay \,|_{C(P)}\, c$ and $ab \,|_{C(P)}\, c$ implies that $by \,|_{C(P)}\, c$. Another application of ternary Pareto gives $cy \,|_{C(P)}\, b$ contrary to $by \,|_{C(P)}\, c$.

The final case is $\{a, b, c\} \cap \{x, y, z\} = \emptyset$. Define $P = (H_1, \ldots, H_k)$ by $H_i = H_{\{a,b,x\},\{c,y,z\}}$ and $H_r = H_{\{x,y,a\},\{x,y,a,c,z\}}$ for all $r \neq i$ in K. Decisiveness yields $ab \,|_{C(P)}\, c$ and $xy \,|_{C(P)}\, z$. By ternary Pareto, $yz \,|_{C(P)}\, b$, $xa \,|_{C(P)}\, z$, and $cy \,|_{C(P)}\, b$. Now $xy \,|_{C(P)}\, z$ and $xa \,|_{C(P)}\, z$ implies that $ay \,|_{C(P)}\, z$. Next, $ay \,|_{C(P)}\, z$ and $yz \,|_{C(P)}\, b$ implies that $ay \,|_{C(P)}\, b$. Observe that $ay \,|_{C(P)}\, b$ and $cy \,|_{C(P)}\, b$ implies that $ac \,|_{C(P)}\, b$ contrary to $ab \,|_{C(P)}\, c$.

For the converse assume that $\{a, b\} = \{x, y\}$. If $c = z$, then define $C : \mathcal{H}^k \to \mathcal{H}$ by

$$C(P) = \begin{cases} H_i & \text{if } ab \,|_{H_i}\, c, \\ H_j & \text{otherwise} \end{cases}$$

for all profiles $P = (H_1, \ldots, H_k)$. In this case, C is ternary Pareto and $\{i\}$ and $\{j\}$ are decisive for (a, b, c). We now consider the case $c \neq z$. For any profile $P = (H_1, \ldots, H_k)$, let

$$\langle a, b\rangle_j = \cap\{X \in H_j : \{a, b\} \subseteq X\}$$

and

$$H_i \,|_{\langle a,b\rangle_j} = \{A \cap \langle a, b\rangle_j : A \in H_i \text{ and } A \cap \langle a, b\rangle_j \neq \emptyset\} \cup H_\emptyset$$

and define $C : \mathcal{H}^k \to \mathcal{H}$ by

$$C(P) = \begin{cases} H_i & \text{if } ab \,|_{H_j}\, z \text{ fails}, \\ H_i \,|_{\langle a,b\rangle_j} \cup \{X \in H_j : X \not\subset \langle a, b\rangle_j\} & \text{otherwise.} \end{cases}$$

It can be checked that C is a ternary Pareto consensus function, $\{i\}$ is decisive for (a, b, c), and $\{j\}$ is decisive for (a, b, z). $\square$

In this paper, we gave an analog of Sen's theorem for consensus functions on hierarchies by showing that in almost all cases it is not possible to find a ternary Pareto consensus function where two distinct algorithms will have the same decisive impact on the consensus output. It would be interesting to find out whether there are ways in which the negative conclusion of Theorem 1 can be avoided. As a first step, one could consult Don Saari's papers [6,7] where he gives a surprisingly elementary explanation of Sen's problem and where he offers some possible resolutions.

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